

# On the $K(\pi, 1)$ -property for rings of integers in the mixed case

by Alexander Schmidt

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## Abstract

We investigate the Galois group  $G_S(p)$  of the maximal  $p$ -extension unramified outside a finite set  $S$  of primes of a number field in the (mixed) case, when there are primes dividing  $p$  inside and outside  $S$ . We show that the cohomology of  $G_S(p)$  is ‘often’ isomorphic to the étale cohomology of the scheme  $\text{Spec}(\mathcal{O}_k \setminus S)$ , in particular,  $G_S(p)$  is of cohomological dimension 2 then. We deduce this from the results in our previous paper [Sch2], which mainly dealt with the tame case.

## 1 Introduction

Let  $Y$  be a connected locally noetherian scheme and let  $p$  be a prime number. We denote the étale fundamental group of  $Y$  by  $\pi_1(Y)$  and its maximal pro- $p$  factor group by  $\pi_1(Y)(p)$ . The Hochschild-Serre spectral sequence induces natural homomorphisms

$$\phi_i : H^i(\pi_1^{et}(Y)(p), \mathbb{Z}/p\mathbb{Z}) \longrightarrow H_{et}^i(Y, \mathbb{Z}/p\mathbb{Z}), \quad i \geq 0,$$

and we call  $Y$  a ‘ $K(\pi, 1)$  for  $p$ ’ if all  $\phi_i$  are isomorphisms; see [Sch2] Proposition 2.1 for equivalent conditions. See [Wi2] for a purely Galois cohomological approach to the  $K(\pi, 1)$ -property. Our main result is the following

**Theorem 1.1.** *Let  $k$  be a number field and let  $p$  be a prime number. Assume that  $k$  does not contain a primitive  $p$ -th root of unity and that the class number of  $k$  is prime to  $p$ . Then the following holds:*

*Let  $S$  be a finite set of primes of  $k$  and let  $T$  be a set of primes of  $k$  of Dirichlet density  $\delta(T) = 1$ . Then there exists a finite subset  $T_1 \subset T$  such that  $\text{Spec}(\mathcal{O}_k) \setminus (S \cup T_1)$  is a  $K(\pi, 1)$  for  $p$ .*

**Remarks.** 1. If  $S$  contains the set  $S_p$  of primes dividing  $p$ , then Theorem 1.1 holds with  $T_1 = \emptyset$  and even without the condition  $\zeta_p \notin k$  and  $Cl(k)(p) = 0$ , see [Sch2], Proposition 2.3. In the tame case  $S \cap S_p = \emptyset$ , the statement of Theorem 1.1 is the main result of [Sch2]. Here we provide the extension to the ‘mixed’ case  $\emptyset \subsetneq S \cap S_p \subsetneq S_p$ .

2. For a given number field  $k$ , all but finitely many prime numbers  $p$  satisfy the condition of Theorem 1.1. We conjecture that Theorem 1.1 holds without the restricting assumption on  $p$ .

Let  $S$  be a finite set of places of a number field  $k$ . Let  $k_S(p)$  be the maximal  $p$ -extension of  $k$  unramified outside  $S$  and put  $G_S(p) = \text{Gal}(k_S(p)|k)$ . If  $S_{\mathbb{R}}$  denotes the set of real places of  $k$ , then  $G_{S \cup S_{\mathbb{R}}}(p) \cong \pi_1(\text{Spec}(\mathcal{O}_k) \setminus S)(p)$  (we have  $G_S(p) = G_{S \cup S_{\mathbb{R}}}(p)$  if  $p$  is odd or  $k$  is totally imaginary). The following Theorem 1.2 sharpens Theorem 1.1.

**Theorem 1.2.** *The set  $T_1 \subset T$  in Theorem 1.1 may be chosen such that*

- (i)  $T_1$  consists of primes  $\mathfrak{p}$  of degree 1 with  $N(\mathfrak{p}) \equiv 1 \pmod{p}$ ,
- (ii)  $(k_{S \cup T_1}(p))_{\mathfrak{p}} = k_{\mathfrak{p}}(p)$  for all primes  $\mathfrak{p} \in S \cup T_1$ .

Note that Theorem 1.2 provides nontrivial information even in the case  $S \supset S_p$ , where assertion (ii) was only known when  $k$  contains a primitive  $p$ -th root of unity (Kuz'min's theorem, see [Kuz] or [NSW], 10.6.4 or [NSW<sup>2</sup>], 10.8.4, respectively) and for certain CM fields (by a result of Mukhamedov, see [Muk] or [NSW], X §6 exercise or [NSW<sup>2</sup>], X §8 exercise, respectively).

By Theorem 3.3 below, Theorem 1.2 provides many examples of  $G_S(p)$  being a duality group. If  $\zeta_p \notin k$ , this is interesting even in the case that  $S \supset S_p$ , where examples of  $G_S(p)$  being a duality group were previously known only for real abelian fields and for certain CM-fields (see [NSW], 10.7.15 and [NSW<sup>2</sup>], 10.9.15, respectively, and the remark following there).

Previous results in the mixed case had been achieved by K. Wingberg [W1], Ch. Maire [Mai] and D. Vogel [Vog]. Though not explicitly visible in this paper, the present progress in the subject was only possible due to the results on mild pro- $p$  groups obtained by J. Labute in [Lab].

I would like to thank K. Wingberg for pointing out that the proof of Proposition 8.1 in my paper [Sch2] did not use the assumption that the sets  $S$  and  $S'$  are disjoint from  $S_p$ . This was the key observation for the present paper. The main part of this text was written while I was a guest at the Department of Mathematical Sciences of Tokyo University and of the Research Institute for Mathematical Sciences in Kyoto. I want to thank these institutions for their kind hospitality.

## 2 Proof of Theorems 1.1 and 1.2

We start with the observation that the proofs of Proposition 8.2 and Corollary 8.3 in [Sch2] did not use the assumption that the sets  $S$  and  $S'$  are disjoint from  $S_p$ . Therefore, with the same proof (which we repeat for the convenience of the reader) as in loc. cit., we obtain

**Proposition 2.1.** *Let  $k$  be a number field and let  $p$  be a prime number. Assume  $k$  to be totally imaginary if  $p = 2$ . Put  $X = \text{Spec}(\mathcal{O}_k)$  and let  $S \subset S'$  be finite sets of primes of  $k$ . Assume that  $X \setminus S$  is a  $K(\pi, 1)$  for  $p$  and that  $G_S(p) \neq 1$ . Further assume that each  $\mathfrak{p} \in S' \setminus S$  does not split completely in  $k_S(p)$ . Then the following hold.*

- (i)  $X \setminus S'$  is a  $K(\pi, 1)$  for  $p$ .
- (ii)  $k_{S'}(p) = k_{\mathfrak{p}}(p)$  for all  $\mathfrak{p} \in S' \setminus S$ .

Furthermore, the arithmetic form of Riemann's existence theorem holds, i.e., setting  $K = k_S(p)$ , the natural homomorphism

$$\bigstar_{\mathfrak{p} \in S' \setminus S(K)} T(K_{\mathfrak{p}}(p)|K_{\mathfrak{p}}) \longrightarrow \text{Gal}(k_{S'}(p)|K)$$

is an isomorphism. Here  $T(K_{\mathfrak{p}}(p)|K_{\mathfrak{p}})$  is the inertia group and  $\bigstar$  denotes the free pro- $p$ -product of a bundle of pro- $p$ -groups, cf. [NSW], Ch. IV, §3. In particular,  $\text{Gal}(k_{S'}(p)|k_S(p))$  is a free pro- $p$ -group.

*Proof.* The  $K(\pi, 1)$ -property implies

$$H^i(G_S(p), \mathbb{Z}/p\mathbb{Z}) \cong H_{et}^i(X \setminus S, \mathbb{Z}/p\mathbb{Z}) = 0 \text{ for } i \geq 4,$$

hence  $cd \ G_S(p) \leq 3$ . Let  $\mathfrak{p} \in S' \setminus S$ . Since  $\mathfrak{p}$  does not split completely in  $k_S(p)$  and since  $cd \ G_S(p) < \infty$ , the decomposition group of  $\mathfrak{p}$  in  $k_S(p)|k$  is a non-trivial and torsion-free quotient of  $\mathbb{Z}_p \cong \text{Gal}(k_{\mathfrak{p}}^{nr}(p)|k_{\mathfrak{p}})$ . Therefore  $k_S(p)_{\mathfrak{p}}$  is the maximal unramified  $p$ -extension of  $k_{\mathfrak{p}}$ . We denote the normalization of an integral normal scheme  $Y$  in an algebraic extension  $L$  of its function field by  $Y_L$ . Then  $(X \setminus S)_{k_S(p)}$  is the universal pro- $p$  covering of  $X \setminus S$ . We consider the étale excision sequence for the pair  $((X \setminus S)_{k_S(p)}, (X \setminus S')_{k_S(p)})$ . By assumption,  $X \setminus S$  is a  $K(\pi, 1)$  for  $p$ , hence  $H_{et}^i((X \setminus S)_{k_S(p)}, \mathbb{Z}/p\mathbb{Z}) = 0$  for  $i \geq 1$  by [Sch2], Proposition 2.1. Omitting the coefficients  $\mathbb{Z}/p\mathbb{Z}$  from the notation, this implies isomorphisms

$$H_{et}^i((X \setminus S')_{k_S(p)}) \xrightarrow{\sim} \bigoplus'_{\mathfrak{p} \in S' \setminus S(k_S(p))} H_{\mathfrak{p}}^{i+1}(((X \setminus S)_{k_S(p)})_{\mathfrak{p}})$$

for  $i \geq 1$ . Here (and in variants also below) we use the notational convention

$$\bigoplus'_{\mathfrak{p} \in S' \setminus S(k_S(p))} H_{\mathfrak{p}}^{i+1}(((X \setminus S)_{k_S(p)})_{\mathfrak{p}}) := \varinjlim_{K \subset k_S(p)} \bigoplus_{\mathfrak{p} \in S' \setminus S(K)} H_{\mathfrak{p}}^{i+1}(((X \setminus S)_K)_{\mathfrak{p}}),$$

where  $K$  runs through the finite extensions of  $k$  inside  $k_S(p)$ . As  $k_S(p)$  realizes the maximal unramified  $p$ -extension of  $k_{\mathfrak{p}}$  for all  $\mathfrak{p} \in S' \setminus S$ , the schemes  $((X \setminus S)_{k_S(p)})_{\mathfrak{p}}$ ,  $\mathfrak{p} \in S' \setminus S(k_S(p))$ , have trivial cohomology with values in  $\mathbb{Z}/p\mathbb{Z}$  and we obtain isomorphisms

$$H^i((k_S(p))_{\mathfrak{p}}) \xrightarrow{\sim} H_{\mathfrak{p}}^{i+1}(((X \setminus S)_{k_S(p)})_{\mathfrak{p}})$$

for  $i \geq 1$ . These groups vanish for  $i \geq 2$ . This implies

$$H_{et}^i((X \setminus S')_{k_S(p)}) = 0$$

for  $i \geq 2$ . Since the scheme  $(X \setminus S')_{k_{S'}(p)}$  is the universal pro- $p$  covering of  $(X \setminus S')_{k_S(p)}$ , the Hochschild-Serre spectral sequence yields an inclusion

$$H^2(Gal(k_{S'}(p)|k_S(p))) \hookrightarrow H_{et}^2((X \setminus S')_{k_S(p)}) = 0.$$

Hence  $Gal(k_{S'}(p)|k_S(p))$  is a free pro- $p$ -group and

$$H^1(Gal(k_{S'}(p)|k_S(p))) \xrightarrow{\sim} H_{et}^1((X \setminus S')_{k_S(p)}) \cong \bigoplus'_{\mathfrak{p} \in S' \setminus S(k_S(p))} H^1(k_S(p)_{\mathfrak{p}}).$$

We set  $K = k_S(p)$  and consider the natural homomorphism

$$\phi : \bigast_{\mathfrak{p} \in S' \setminus S(K)} T(K_{\mathfrak{p}}(p)|K_{\mathfrak{p}}) \longrightarrow Gal(k_{S'}(p)|K).$$

By the calculation of the cohomology of a free product ([NSW], 4.3.10 and 4.1.4),  $\phi$  is a homomorphism between free pro- $p$ -groups which induces an isomorphism on mod  $p$  cohomology. Therefore  $\phi$  is an isomorphism. In particular,  $k_{S'}(p)_{\mathfrak{p}} = k_{\mathfrak{p}}(p)$  for all  $\mathfrak{p} \in S' \setminus S$ . Using that  $Gal(k_{S'}(p)|k_S(p))$  is free, the Hochschild-Serre spectral sequence

$$E_2^{ij} = H^i(Gal(k_{S'}(p)|k_S(p)), H_{et}^j((X \setminus S')_{k_{S'}(p)})) \Rightarrow H_{et}^{i+j}((X \setminus S')_{k_S(p)})$$

induces an isomorphism

$$0 = H_{et}^2((X \setminus S')_{k_S(p)}) \xrightarrow{\sim} H_{et}^2((X \setminus S')_{k_{S'}(p)})^{Gal(k_{S'}(p)|k_S(p))}.$$

Hence  $H_{et}^2((X \setminus S')_{k_{S'}(p)}) = 0$ , since  $Gal(k_{S'}(p)|k_S(p))$  is a pro- $p$ -group. Now [Sch2], Proposition 2.1 implies that  $X \setminus S'$  is a  $K(\pi, 1)$  for  $p$ .  $\square$

In order to prove Theorem 1.1, we first provide the following lemma. For an extension field  $K|k$  and a set of primes  $T$  of  $k$ , we write  $T(K)$  for the set of prolongations of primes in  $T$  to  $K$  and  $\delta_K(T)$  for the Dirichlet density of the set of primes  $T(K)$  of  $K$ .

**Lemma 2.2.** *Let  $k$  be a number field,  $p$  a prime number and  $S$  a finite set of nonarchimedean primes of  $k$ . Let  $T$  be a set of primes of  $k$  with  $\delta_{k(\mu_p)}(T) = 1$ . Then there exists a finite subset  $T_0 \subset T$  such that all primes  $\mathfrak{p} \in S$  do not split completely in the extension  $k_{T_0}(p)|k$ .*

*Proof.* By [NSW], 9.2.2 (ii) or [NSW<sup>2</sup>], 9.2.3 (ii), respectively, the restriction map

$$H^1(G_{T \cup S \cup S_p \cup S_{\mathbb{R}}}(p), \mathbb{Z}/p\mathbb{Z}) \longrightarrow \prod_{\mathfrak{p} \in S \cup S_p \cup S_{\mathbb{R}}} H^1(k_{\mathfrak{p}}, \mathbb{Z}/p\mathbb{Z})$$

is surjective. A class in  $\alpha \in H^1(G_{T \cup S \cup S_p \cup S_{\mathbb{R}}}(p), \mathbb{Z}/p\mathbb{Z})$  which restricts to an unramified class  $\alpha_{\mathfrak{p}} \in H_{nr}^1(k_{\mathfrak{p}}, \mathbb{Z}/p\mathbb{Z})$  for all  $\mathfrak{p} \in S \cup S_p \cup S_{\mathbb{R}}$  is contained in  $H^1(G_T(p), \mathbb{Z}/p\mathbb{Z})$ . Therefore the image of the composite map

$$H^1(G_T(p), \mathbb{Z}/p\mathbb{Z}) \hookrightarrow H^1(G_{T \cup S \cup S_p \cup S_{\mathbb{R}}}(p), \mathbb{Z}/p\mathbb{Z}) \rightarrow \prod_{\mathfrak{p} \in S} H^1(k_{\mathfrak{p}}, \mathbb{Z}/p\mathbb{Z})$$

contains the subgroup  $\prod_{\mathfrak{p} \in S} H_{nr}^1(k_{\mathfrak{p}}, \mathbb{Z}/p\mathbb{Z})$ . As this group is finite, it is already contained in the image of  $H^1(G_{T_0}(p), \mathbb{Z}/p\mathbb{Z})$  for some finite subset  $T_0 \subset T$ . We conclude that no prime in  $S$  splits completely in the maximal elementary abelian  $p$ -extension of  $k$  unramified outside  $T_0$ .  $\square$

*Proof of Theorems 1.1 and 1.2.* As  $p \neq 2$ , we may ignore archimedean primes. Furthermore, we may remove the primes in  $S \cup S_p$  and all primes of degree greater than 1 from  $T$ . In addition, we remove all primes  $\mathfrak{p}$  with  $N(\mathfrak{p}) \not\equiv 1 \pmod p$  from  $T$ . After these changes, we still have  $\delta_{k(\mu_p)}(T) = 1$ .

By Lemma 2.2, we find a finite subset  $T_0 \subset T$  such that no prime in  $S$  splits completely in  $k_{T_0}(p)|k$ . Put  $X = \text{Spec}(\mathcal{O}_k)$ . By [Sch2], Theorem 6.2, applied to  $T_0$  and  $T \setminus T_0$ , we find a finite subset  $T_2 \subset T \setminus T_0$  such that  $X \setminus (T_0 \cup T_2)$  is a  $K(\pi, 1)$  for  $p$ . Then Proposition 2.1 applied to  $T_0 \cup T_2 \subset S \cup T_0 \cup T_2$ , shows that also  $X \setminus (S \cup T_0 \cup T_2)$  is a  $K(\pi, 1)$  for  $p$ . Now put  $T_1 = T_0 \cup T_2 \subset T$ .

It remains to show Theorem 1.2. Assertion (i) holds by construction of  $T_1$ . By [Sch2], Lemma 4.1, also  $X \setminus (S' \cup T_1)$  is a  $K(\pi, 1)$  for  $p$ . By [Sch2], Theorem 3, the field  $k_{T_1}(p)$  realizes  $k_{\mathfrak{p}}(p)$  for  $\mathfrak{p} \in T_1$ , showing (ii) for these primes. Finally, assertion (ii) for  $\mathfrak{p} \in S$  follows from Proposition 2.1.  $\square$

### 3 Duality

We start by investigating the relation between the  $K(\pi, 1)$ -property and the universal norms of global units.

Let us first remove redundant primes from  $S$ : If  $\mathfrak{p} \nmid p$  is a prime with  $\zeta_p \notin k_{\mathfrak{p}}$ , then every  $p$ -extension of the local field  $k_{\mathfrak{p}}$  is unramified (see [NSW], 7.5.1 or [NSW<sup>2</sup>], 7.5.9, respectively). Therefore primes  $\mathfrak{p} \notin S_p$  with  $N(\mathfrak{p}) \not\equiv 1 \pmod p$  cannot ramify in a  $p$ -extension. Removing all these redundant primes from  $S$ , we obtain a subset  $S_{\min} \subset S$ , which has the property that  $G_S(p) = G_{S_{\min}}(p)$ . Furthermore, by [Sch2], Lemma 4.1,  $X \setminus S$  is a  $K(\pi, 1)$  for  $p$  if and only if  $X \setminus S_{\min}$  is a  $K(\pi, 1)$  for  $p$ .

**Theorem 3.1.** *Let  $k$  be a number field and let  $p$  be a prime number. Assume that  $k$  is totally imaginary if  $p = 2$ . Let  $S$  be a finite set of nonarchimedean primes of  $k$ . Then any two of the following conditions (a) – (c) imply the third.*

- (a)  $\text{Spec}(\mathcal{O}_k) \setminus S$  is a  $K(\pi, 1)$  for  $p$ .
- (b)  $\varprojlim_{K \subset k_S(p)} \mathcal{O}_K^{\times} \otimes \mathbb{Z}_p = 0$ .
- (c)  $(k_S(p))_{\mathfrak{p}} = k_{\mathfrak{p}}(p)$  for all primes  $\mathfrak{p} \in S_{\min}$ .

The limit in (b) runs through all finite extensions  $K$  of  $k$  inside  $k_S(p)$ . If (a)–(c) hold, then also

$$\varprojlim_{K \subset k_S(p)} \mathcal{O}_{K, S_{\min}}^{\times} \otimes \mathbb{Z}_p = 0.$$

**Remarks:** 1. Assume that  $\zeta_p \in k$  and  $S \supset S_p$ . Then (a) holds and condition (b) holds for  $p > 2$  if  $\#S > r_2 + 2$  (see [NSW<sup>2</sup>], Remark 2 after 10.9.3). In the

case  $k = \mathbb{Q}(\zeta_p)$ ,  $S = S_p$ , condition (b) holds if and only if  $p$  is an irregular prime number.

2. Assume that  $S \cap S_p = \emptyset$  and  $S_{\min} \neq \emptyset$ . If condition (a) holds, then either  $G_S(p) = 1$  (which only happens in very special situations, see [Sch2], Proposition 7.4) or (b) holds by [Sch2], Theorem 3 (or by Proposition 3.2 below).

*Proof of Theorem 3.1.* We may assume  $S = S_{\min}$  in the proof. Let  $K$  run through the finite extensions of  $k$  in  $k_S(p)$  and put  $X_K = \text{Spec}(\mathcal{O}_K)$ . Applying the topological Nakayama-Lemma ([NSW], 5.2.18) to the compact  $\mathbb{Z}_p$ -module  $\varprojlim \mathcal{O}_K^\times \otimes \mathbb{Z}_p$ , we see that condition (b) is equivalent to

$$(b)' \quad \varprojlim_{K \subset k_S(p)} \mathcal{O}_K^\times / p = 0.$$

Furthermore, by [Sch2], Proposition 2.1, condition (a) is equivalent to

$$(a)' \quad \varinjlim_{K \subset k_S(p)} H_{et}^i((X \setminus S)_K, \mathbb{Z}/p\mathbb{Z}) = 0 \text{ for } i \geq 1.$$

Condition (a)' always holds for  $i = 1$ ,  $i \geq 4$ , and it holds for  $i = 3$  provided that  $G_S(p)$  is infinite or  $S$  is nonempty or  $\zeta_p \notin k$  (see [Sch2], Lemma 3.7). The flat Kummer sequence  $0 \rightarrow \mu_p \rightarrow \mathbb{G}_m \xrightarrow{p} \mathbb{G}_m \rightarrow 0$  induces exact sequences

$$0 \longrightarrow \mathcal{O}_K^\times / p \longrightarrow H_{\text{fl}}^1(X_K, \mu_p) \longrightarrow {}_p\text{Pic}(X) \rightarrow 0$$

for all  $K$ . As the field  $k_S(p)$  does not have nontrivial unramified  $p$ -extensions, class field theory implies

$$\varprojlim_{K \subset k_S(p)} {}_p\text{Pic}(X_K) \subset \varprojlim_{K \subset k_S(p)} \text{Pic}(X_K) \otimes \mathbb{Z}_p = 0.$$

As we assumed  $k$  to be totally imaginary if  $p = 2$ , the flat duality theorem of Artin-Mazur ([Mil], III Corollary 3.2) induces natural isomorphisms

$$H_{et}^2(X_K, \mathbb{Z}/p\mathbb{Z}) = H_{\text{fl}}^2(X_K, \mathbb{Z}/p\mathbb{Z}) \cong H_{\text{fl}}^1(X_K, \mu_p)^\vee.$$

We conclude that

$$(*) \quad \varinjlim_{K \subset k_S(p)} H_{et}^2(X_K, \mathbb{Z}/p\mathbb{Z}) \cong \left( \varprojlim_{K \subset k_S(p)} \mathcal{O}_K^\times / p \right)^\vee.$$

We first show the equivalence of (a) and (b) in the case  $S = \emptyset$ . If (a)' holds, then (\*) shows (b)'. If (b) holds, then  $\zeta_p \notin k$  or  $G_S(p)$  is infinite. Hence we obtain (a)' for  $i = 3$ . Furthermore, (b)' implies (a)' for  $i = 2$  by (\*). This finishes the proof of the case  $S = \emptyset$ .

Now we assume that  $S \neq \emptyset$ . For  $\mathfrak{p} \in S(K)$ , a standard calculation of local cohomology shows that

$$H_{\mathfrak{p}}^i(X_K, \mathbb{Z}/p\mathbb{Z}) \cong \begin{cases} 0 & \text{for } i \leq 1, \\ H^1(K_{\mathfrak{p}}, \mathbb{Z}/p\mathbb{Z}) / H_{nr}^1(K_{\mathfrak{p}}, \mathbb{Z}/p\mathbb{Z}) & \text{for } i = 2, \\ H^2(K_{\mathfrak{p}}, \mathbb{Z}/p\mathbb{Z}) & \text{for } i = 3. \\ 0 & \text{for } i \geq 4. \end{cases}$$

For  $\mathfrak{p} \in S = S_{\min}$ , every proper Galois subextension of  $k_{\mathfrak{p}}(p)|k_{\mathfrak{p}}$  admits ramified  $p$ -extensions. Hence condition (c) is equivalent to

$$(c)' \quad \varinjlim_{K \subset k_S(p)} \bigoplus_{\mathfrak{p} \in S(K)} H_{\mathfrak{p}}^i(X_K, \mathbb{Z}/p\mathbb{Z}) = 0 \text{ for all } i,$$

and to

$$(c)'' \quad \varinjlim_{K \subset k_S(p)} \bigoplus_{\mathfrak{p} \in S(K)} H_{\mathfrak{p}}^2(X_K, \mathbb{Z}/p\mathbb{Z}) = 0.$$

Consider the direct limit over all  $K$  of the excision sequences

$$\cdots \rightarrow \bigoplus_{\mathfrak{p} \in S(K)} H_{\mathfrak{p}}^i(X_K, \mathbb{Z}/p\mathbb{Z}) \rightarrow H_{et}^i(X_K, \mathbb{Z}/p\mathbb{Z}) \rightarrow H_{et}^i((X \setminus S)_K, \mathbb{Z}/p\mathbb{Z}) \rightarrow \cdots.$$

Assume that (a)' holds, i.e. the right hand terms vanish in the limit for  $i \geq 1$ . Then (\*) shows that (b)' is equivalent to (c)'.

Now assume that (b) and (c) hold. As above, (b) implies the vanishing of the middle term for  $i = 2, 3$  in the limit. Condition (c)' then shows (a)'.

We have proven that any two of the conditions (a)–(c) imply the third.

Finally, assume that (a)–(c) hold. Tensoring the exact sequences (cf. [NSW], 10.3.11 or [NSW<sup>2</sup>], 10.3.12, respectively)

$$0 \rightarrow \mathcal{O}_K^{\times} \rightarrow \mathcal{O}_{K,S}^{\times} \rightarrow \bigoplus_{\mathfrak{p} \in S(K)} (K_{\mathfrak{p}}^{\times}/U_{\mathfrak{p}}) \rightarrow \text{Pic}(X_K) \rightarrow \text{Pic}((X \setminus S)_K) \rightarrow 0$$

by (the flat  $\mathbb{Z}$ -algebra)  $\mathbb{Z}_p$ , we obtain exact sequences of finitely generated, hence compact,  $\mathbb{Z}_p$ -modules. Passing to the projective limit over the finite extensions  $K$  of  $k$  inside  $k_S(p)$  and using  $\varprojlim \text{Pic}(X_K) \otimes \mathbb{Z}_p = 0$ , we obtain the exact sequence

$$0 \rightarrow \varprojlim_{K \subset k_S(p)} \mathcal{O}_K^{\times} \otimes \mathbb{Z}_p \rightarrow \varprojlim_{K \subset k_S(p)} \mathcal{O}_{K,S}^{\times} \otimes \mathbb{Z}_p \rightarrow \varprojlim_{K \subset k_S(p)} \bigoplus_{\mathfrak{p} \in S(K)} (K_{\mathfrak{p}}^{\times}/U_{\mathfrak{p}}) \otimes \mathbb{Z}_p \rightarrow 0.$$

Condition (c) and local class field theory imply the vanishing of the right hand limit. Therefore (b) implies the vanishing of the projective limit in the middle.  $\square$

If  $G_S(p) \neq 1$  and condition (a) of Theorem 1.1 holds, then the failure in condition (c) can only come from primes dividing  $p$ . This follows from the next

**Proposition 3.2.** *Let  $k$  be a number field and let  $p$  be a prime number. Assume that  $k$  is totally imaginary if  $p = 2$ . Let  $S$  be a finite set of nonarchimedean primes of  $k$ . If  $\text{Spec}(\mathcal{O}_k) \setminus S$  is a  $K(\pi, 1)$  for  $p$  and  $G_S(p) \neq 1$ , then every prime  $\mathfrak{p} \in S$  with  $\zeta_p \in k_{\mathfrak{p}}$  has an infinite inertia group in  $G_S(p)$ . Moreover, we have*

$$k_S(p)_{\mathfrak{p}} = k_{\mathfrak{p}}(p)$$

for all  $\mathfrak{p} \in S_{\min} \setminus S_p$ .

*Proof.* We may assume  $S = S_{\min}$ . Suppose  $\mathfrak{p} \in S$  with  $\zeta_p \in k_{\mathfrak{p}}$  does not ramify in  $k_S(p)|k$ . Setting  $S' = S \setminus \{\mathfrak{p}\}$ , we have  $k_{S'}(p) = k_S(p)$ , in particular,

$$H_{et}^1(X \setminus S', \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} H_{et}^1(X \setminus S, \mathbb{Z}/p\mathbb{Z}).$$

In the following, we omit the coefficients  $\mathbb{Z}/p\mathbb{Z}$  from the notation. Using the vanishing of  $H_{et}^3(X \setminus S)$ , the étale excision sequence yields a commutative exact diagram

$$\begin{array}{ccccccc} & & H^2(G_{S'}(p)) & \xrightarrow{\sim} & H^2(G_S(p)) & & \\ & & \downarrow & & \downarrow \wr & & \\ H_{\mathfrak{p}}^2(X) & \hookrightarrow & H_{et}^2(X \setminus S') & \xrightarrow{\alpha} & H_{et}^2(X \setminus S) & \longrightarrow & H_{\mathfrak{p}}^3(X) \twoheadrightarrow H_{et}^3(X \setminus S'). \end{array}$$

Hence  $\alpha$  is split-surjective and  $\mathbb{Z}/p\mathbb{Z} \cong H_{\mathfrak{p}}^3(X) \xrightarrow{\sim} H_{et}^3(X \setminus S')$ . This implies  $S' = \emptyset$ , hence  $S = \{\mathfrak{p}\}$ , and  $\zeta_p \in k$ . The same applies to every finite extension of  $k$  in  $k_S(p)$ , hence  $\mathfrak{p}$  is inert in  $k_S(p) = k_{\emptyset}(p)$ . This implies that the natural homomorphism

$$Gal(k_{\mathfrak{p}}^{nr}(p)|k_{\mathfrak{p}}) \longrightarrow G_{\emptyset}(k)(p)$$

is surjective. Therefore  $G_S(p) = G_{\emptyset}(p)$  is abelian, hence finite by class field theory. Since this group has finite cohomological dimension by the  $K(\pi, 1)$ -property, it is trivial, in contradiction to our assumptions.

This shows that all  $\mathfrak{p} \in S$  with  $\zeta_p \in k_{\mathfrak{p}}$  ramify in  $k_S(p)$ . As this applies to every finite extension of  $k$  inside  $k_S(p)$ , the inertia groups must be infinite. For  $\mathfrak{p} \in S_{\min} \setminus S_p$  this implies  $k_S(p)_{\mathfrak{p}} = k_{\mathfrak{p}}(p)$ .  $\square$

**Theorem 3.3.** *Let  $k$  be a number field and let  $p$  be a prime number. Assume that  $k$  is totally imaginary if  $p = 2$ . Let  $S$  be a finite nonempty set of nonarchimedean primes of  $k$ . Assume that conditions (a)–(c) of Theorem 3.1 hold and that  $\zeta_p \in k_{\mathfrak{p}}$  for all  $\mathfrak{p} \in S$ . Then  $G_S(p)$  is a pro- $p$  duality group of dimension 2.*

*Proof.* Condition (a) implies  $H^3(G_S(p), \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} H_{et}^3(X \setminus S, \mathbb{Z}/p\mathbb{Z}) = 0$ . Hence  $cd G_S(p) \leq 2$ . On the other hand, by (c), the group  $G_S(p)$  contains  $Gal(k_{\mathfrak{p}}(p)|k_{\mathfrak{p}})$  as a subgroup for all  $\mathfrak{p} \in S$ . As  $\zeta_p \in k_{\mathfrak{p}}$  for  $\mathfrak{p} \in S$ , these local groups have cohomological dimension 2, hence so does  $G_S(p)$ .

In order to show that  $G_S(p)$  is a duality group, we have to show that

$$D_i(G_S(p), \mathbb{Z}/p\mathbb{Z}) := \varinjlim_{\substack{U \subset G_S(p) \\ \text{cor}^\vee}} H^i(U, \mathbb{Z}/p\mathbb{Z})^\vee$$

vanish for  $i = 0, 1$ , where  $U$  runs through the open subgroups of  $G_S(p)$  and the transition maps are the duals of the corestriction homomorphisms; see [NSW], 3.4.6. The vanishing of  $D_0$  is obvious, as  $G_S(p)$  is infinite. Using (a), we therefore have to show that

$$\varinjlim_{K \subset k_S(p)} H^1((X \setminus S)_K, \mathbb{Z}/p\mathbb{Z})^\vee = 0.$$



We put  $X = \text{Spec}(\mathcal{O}_k)$  and denote the embedding by  $j : (X \setminus S)_K \rightarrow X_K$ . By the flat duality theorem of Artin-Mazur, we have natural isomorphisms

$$H^1((X \setminus S)_K, \mathbb{Z}/p\mathbb{Z})^\vee \cong H_{\text{fl},c}^2((X \setminus S)_K, \mu_p) = H_{\text{fl}}^2(X_K, j_! \mu_p).$$

The excision sequence together with a straightforward calculation of local cohomology groups shows an exact sequence

$$(*) \quad \bigoplus_{\mathfrak{p} \in S(K)} K_{\mathfrak{p}}^\times / K_{\mathfrak{p}}^{\times p} \rightarrow H_{\text{fl}}^2(X_K, j_! \mu_p) \rightarrow H_{\text{fl}}^2((X \setminus S)_K, \mu_p).$$

As  $\zeta_p \in k_{\mathfrak{p}}$  and  $k_S(p)_{\mathfrak{p}} = k_{\mathfrak{p}}(p)$  for  $\mathfrak{p} \in S$  by assumption, the left hand term of  $(*)$  vanishes when passing to the limit over all  $K$ . We use the Kummer sequence to obtain an exact sequence

$$(**) \quad \text{Pic}((X \setminus S)_K)/p \longrightarrow H_{\text{fl}}^2((X \setminus S)_K, \mu_p) \longrightarrow {}_p\text{Br}((X \setminus S)_K).$$

The left hand term of  $(**)$  vanishes in the limit by the principal ideal theorem. The Hasse principle for the Brauer group induces an injection

$${}_p\text{Br}((X \setminus S)_K) \hookrightarrow \bigoplus_{\mathfrak{p} \in S(K)} {}_p\text{Br}(K_{\mathfrak{p}}).$$

As  $k_S(p)$  realizes the maximal unramified  $p$ -extension of  $k_{\mathfrak{p}}$  for  $\mathfrak{p} \in S$ , the limit of the middle term in  $(**)$ , and hence also the limit of the middle term in  $(*)$  vanishes. This shows that  $G_S(p)$  is a duality group of dimension 2.  $\square$

**Remark:** The dualizing module can be calculated to

$$D \cong \text{tor}_p(C_S(k_S(p))),$$

i.e.  $D$  is isomorphic to the  $p$ -torsion subgroup in the  $S$ -idèle class group of  $k_S(p)$ . The proof is the same as in ([Sch1], Proof of Thm. 5.2), where we dealt with the tame case.

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Alexander Schmidt, NWF I - Mathematik, Universität Regensburg, D-93040 Regensburg, Deutschland. email: [alexander.schmidt@mathematik.uni-regensburg.de](mailto:alexander.schmidt@mathematik.uni-regensburg.de)